

A PROOF OF A CONJECTURE OF BUCK, CHAN AND ROBBINS ON THE RANDOM ASSIGNMENT PROBLEM

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ABSTRACT. We prove the main conjecture of the paper “On the expected value of the minimum assignment” by Marshall W. Buck, Clara S. Chan, and David P. Robbins (Random Structures & Algorithms 21 (2002), no. 1, 33–58). This is a vast generalization of a formula conjectured by Giorgio Parisi for the n by n random assignment problem.

1. THE PARISI FORMULA

This work is motivated by a conjecture made in 1998 by the physicist Giorgio Parisi [P98]. Consider an n by n matrix of independent $\exp(1)$ random variables. Parisi conjectured that the expected value of the minimal sum of n matrix entries of which no two belong to the same row or column, is given by the formula

$$(1) \quad \sum_{i=1}^n \frac{1}{i^2}.$$

An equivalent setting is obtained by considering the expected minimum cost of a perfect matching in a complete n by n bipartite graph with independent $\exp(1)$ edge costs.

The problem had already received quite some attention. At the time, the main open question was the value of the limit of the expected optimal value as n tends to infinity. A non-rigorous argument due to Marc Mézard and Parisi [MP85] showed that the limit ought to be $\zeta(2) = \pi^2/6$. David Aldous [A92] proved, using an infinite model, that the limit exists. The striking conjecture of Parisi (obviously consistent with the conjectured $\zeta(2)$ -limit) paved the way for an entirely new approach. It seemed likely that (1) would yield to an inductive argument, and that therefore the $\zeta(2)$ -limit could be established by exact analysis of the “finite n ” case. The Parisi formula was almost immediately generalized by Don Coppersmith and Gregory B. Sorkin [CS98] to

$$(2) \quad \sum_{\substack{i,j \geq 0 \\ i+j < k}} \frac{1}{(m-i)(n-j)}$$

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for k -assignments in an m by n matrix, $k \leq \min(m, n)$. It is not hard to verify that (2) specializes to (1) when $k = m = n$.

The Coppersmith-Sorkin conjecture was then generalized in two different directions by Marshall W. Buck, Clara S. Chan, and David P. Robbins [BCR02] and by the present authors [LW00]. In a recent paper [LW03], we prove our own conjecture, thereby establishing the Parisi and Coppersmith-Sorkin formulas. Remarkably, a proof of these formulas was announced simultaneously by Chandra Nair, Balaji Prabhakar and Mayank Sharma [NPS03].

In this paper we state and prove a simultaneous generalization of the Buck-Chan-Robbins conjecture and the main theorem of [LW03]. Not only is this a stronger result than that of [LW03], but it also provides a considerable improvement of the proof. By combining the approach of [BCR02] with that of [LW03], we obtain a proof of Parisi's formula which is shorter, simpler, and gives a far better insight into the problem than our original proof.

2. THE BUCK-CHAN-ROBBINS FORMULA

In this section, we state the formula conjectured by Buck, Chan and Robbins in [BCR02]. This is a “combinatorial” formula, involving a binomial coefficient. As is shown in [BCR02], there is an equivalent “probabilistic” version of the formula, and one can pass between the two via elementary properties of the Möbius function. In this paper we work in the probabilistic setting. In order to prove our main theorem, Theorem 9.1, it is therefore not necessary to take the detour through the combinatorial formulas. The purpose of this section is therefore mainly to introduce some notation and provide the background. We remark, however, that both in [LW00] and [BCR02], the discovery of the probabilistic formulas was made through formal manipulation of the combinatorial formulas. Therefore the latter have played an important role in obtaining the results of this paper.

Let M be an m by n matrix of nonnegative real numbers. The rows and columns of the matrix will be indexed by *weighted sets* R and C respectively. We may take $R = \{1, \dots, m\}$ and $C = \{1, \dots, n\}$, but the sets R and C come with weight functions w_R and w_C respectively that associate a positive weight to each element of the set. A k -assignment is a set $\pi \subseteq R \times C$ of k matrix positions, or *sites*, of which no two belong to the same row or column. An assignment will also be called an *independent set*. The *cost* of π is the sum

$$\text{cost}_M(\pi) = \sum_{(i,j) \in \pi} M(i,j)$$

of the matrix entries in π . We let $\min_k(M)$ denote the minimum cost of all k -assignments in M .

In [BCR02], the Parisi and Coppersmith-Sorkin conjectures are generalized to a certain type of matrix with entries which are exponential random variables, but not necessarily with parameter 1. We say that a random variable x is *exponential of rate α* if $Pr(x > t) = e^{-\alpha t}$ for every $t \geq 0$. In this case we write $x \sim \exp(\alpha)$. Buck, Chan and Robbins considered the following type of matrix: For every $(i, j) \in R \times C$, $M(i, j)$ is $\exp(w_R(i)w_C(j))$ -distributed, and independent of all other matrix entries. To state the formula, we use the following notation: If X is a set of rows, we let $w_R(X) = \sum_{i \in X} w_R(i)$, and $w_R(\bar{X}) = w_R(R \setminus X)$. We use similar notation for sets of columns.

Theorem 2.1 (Conjectured by Buck, Chan and Robbins 2000). *Let M be a matrix as described above. Then*

$$(3) \quad E[\min_k(M)] = \sum_{\substack{X \subseteq R \\ Y \subseteq C}} \binom{m+n-1-|X|-|Y|}{k-1-|X|-|Y|} \frac{(-1)^{k-1-|X|-|Y|}}{w_R(\bar{X})w_C(\bar{Y})}.$$

Notice that in order for the binomial coefficient to be nonzero, we must have $|X| + |Y| < k$, which resembles the condition $i + j < k$ in the Coppersmith-Sorkin formula (2). It is still not entirely obvious that (3) specializes to the Coppersmith-Sorkin formula when the row- and column weights are set to 1. However, in [BCR02], the formula (3) is shown to be equivalent to a formula given by an urn model. We will generalize this to a setting where a certain set of matrix entries are set to zero.

3. MAIN THEOREM

The main theorem of [LW03] is a formula for the expected value of the minimal k -assignment in a matrix where a specified set of entries are set to zero, and the remaining entries are independent $\exp(1)$ -variables. In this article we prove a formula for the common generalization of the matrices considered in [BCR02] and in [LW03]. We say that M is a *standard matrix* if the entries in a certain set Z of sites are zero, and the remaining entries are independent and distributed according to the row- and column weights, that is, $M(i, j) \sim \exp(w_R(i)w_C(j))$. This is an obvious generalization of the concept of standard matrix in [LW03]. As in [LW00, LW03, BCR02] we give two seemingly different but equivalent formulations of our main theorem.

Let $Z \subseteq R \times C$ be a set of sites. A *file* is a row or a column. Let λ be a set of files. We say that λ is a *cover* of Z if every site in Z lies in a file that belongs to λ . By a cover of the matrix M we mean a cover of the set of zeros of M . By a $k-1$ -*cover* we mean a cover consisting of $k-1$ files. Finally by a *partial $k-1$ -cover* we mean a subset of a $k-1$ -cover.

Let $J_k(M)$ be the set of partial $k-1$ -covers of the zeros of M . Let $\hat{J}_k(M)$ denote the poset consisting of $J_k(M)$ ordered by inclusion,

together with an artificial largest element $\hat{1}$. The $k - 1$ -covers are coatoms in $\hat{J}_k(M)$. Let μ denote the Möbius function on intervals in $\hat{J}_k(M)$ (see e.g. [S] for the basics of Möbius functions).

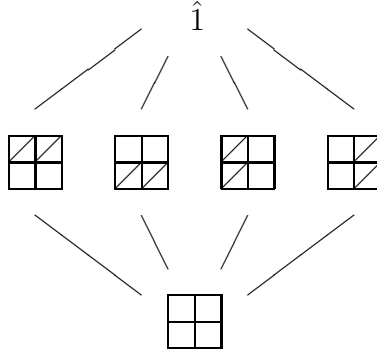
The following is a combinatorial formulation of our main theorem.

Theorem 3.1 (Main Theorem, combinatorial version). *Let M be a standard matrix. Then*

$$(4) \quad E[\min_k(M)] = \sum_{(X,Y) \in J_k(M)} \frac{-\mu((X,Y), \hat{1})}{w_R(\overline{X})w_C(\overline{Y})}.$$

If there are no zero entries in M , $J_k(M)$ consists of all sets of at most $k - 1$ files. The poset $\hat{J}_k(M)$ is a truncated Boolean lattice obtained by deleting all elements of rank $\geq k$ except the top element in the Boolean lattice B_{m+n} . The fact that (4) specializes to (3) follows from the fact that the Möbius function of the truncated Boolean lattice occurring in (4) is given by the signed binomial coefficient in (3).

Example 3.2. Let M be a standard 2×2 matrix with no zeros, and let the row- and column weights be $w_R(i) = a_i$, and $w_C(j) = b_j$. With $k = 2$, the poset $\hat{J}_2(M)$ consists of six elements: The bottom element is the empty set. There are four elements of rank 1 consisting of one file, and then there is the top element $\hat{1}$. The Möbius function on the interval $(\emptyset, \hat{1})$ is equal to 3, and the Möbius function on the intervals from the rank 1 elements to the top element is -1 .



Hence according to (4)

$$(5) \quad E[\min_2(M)] = \frac{-3}{(a_1 + a_2)(b_1 + b_2)} + \frac{1}{a_2(b_1 + b_2)} + \frac{1}{a_1(b_1 + b_2)} + \frac{1}{b_2(a_1 + a_2)} + \frac{1}{b_1(a_1 + a_2)}.$$

If we set the weights equal to 1, (5) specializes to $5/4$, in accordance with the Parisi formula $1 + 1/4$. If we compare to (3), we see that the numerators in (5) are indeed equal to the binomial coefficients in (3)

4. MATRIX REDUCTION

Polynomial time algorithms for computing $\min_k(M)$ for a given (non-random) matrix M are well-known. We do not focus here on issues of computational efficiency, but we outline an algorithm whose special features will be of importance. The following lemma expresses a fundamental property of optimal assignments. It is proved in [BCR02], although these authors make no claims of originality. The first statement is certainly well-known, but we haven't been able to trace the second statement to any other source than [BCR02]. For completeness, we include an outline of the proof.

Lemma 4.1 (Nesting Lemma). *Let M be a real m by n matrix, and let $k_1 \leq k_2 \leq \min(m, n)$ be positive integers. If μ is an optimal k_1 -assignment in M , then there is an optimal k_2 -assignment μ' in M such that every file that intersects μ also intersects μ' . Moreover, if ν is an optimal k_2 -assignment, then there is an optimal k_1 -assignment ν' such that every file that intersects ν' intersects ν .*

Sketch of proof. We may assume that $k_1 = k_2 - 1$. Suppose that μ is an optimal k_1 -assignment, and ν is an optimal k_2 -assignment. Consider the symmetric difference $\delta = \mu \Delta \nu$. We say that two sites are adjacent if they are in the same row or column. The components of δ with respect to adjacency are cycles or paths. Suppose that δ' is a subset of δ which consists of a number of entire components of δ , in other words such that no site in δ' is adjacent to a site in $\delta \setminus \delta'$. Suppose moreover that δ' is balanced in the sense that it contains equally many sites from μ and ν . Then $\mu \Delta \delta'$ is a k_1 -assignment, and $\nu \Delta \delta'$ is a k_2 -assignment. It follows that $\text{cost}(\mu \cap \delta') = \text{cost}(\nu \cap \delta')$. Hence in the components of δ that are balanced, the cost of the sites in μ is equal to the cost of the sites in ν , and for all the other components, the difference in cost between the sites in μ and the sites in ν is the same (with a sign depending on which one of μ and ν is overrepresented). As a consequence, we may take a single component δ_1 of δ so that δ_1 has one more site in ν than in μ . Then $\mu' = \mu \Delta \delta_1$ is an optimal k_2 -assignment, and $\nu' = \nu \Delta \delta_1$ is an optimal k_1 -assignment, and it is straightforward to verify that μ' and ν' have the desired properties. \square

Let $Z \subseteq R \times C$ be a set of sites. We say that a cover of Z is *optimal* if it has the minimum number of files among all covers of Z . The *rank* of a set of sites is the size of the largest independent subset. The following is a famous theorem due to Denes König:

Theorem 4.2 (König's theorem). *The number of files in an optimal cover of Z is equal to $\text{rank}(Z)$.*

The following theorem forms the basis of an algorithm for computing $\min_k(M)$:

Theorem 4.3. *Let M be a nonnegative m by n matrix, and let λ be an optimal cover of M . Suppose that there is no zero cost $k+1$ -assignment in M . Then every file in λ intersects every optimal k -assignment in M .*

Proof. Let μ be an optimal k -assignment. Since there is no zero cost $k+1$ -assignment, $k \geq |\lambda|$. By Lemma 4.1, there is an optimal $|\lambda|$ -assignment μ' such that μ intersects every file that intersects μ' . By König's theorem, there is a zero cost $|\lambda|$ -assignment, and since μ' is optimal, this means that μ' has zero cost. Hence every file in λ intersects μ' . The statement follows. \square

The following matrix operation is fundamental for the algorithm. We refer to it as *matrix reduction*. Let M be a nonnegative m by n matrix, and let $\lambda = X \cup Y$ be an optimal cover of M , where X is the set of rows and Y is the set of columns in λ . The *reduction* M' of M by λ is obtained from M as follows: Let t be the minimum matrix entry of M which is not covered by λ . If the site (i, j) is not covered by λ , we let $M'(i, j) = M(i, j) - t$. In particular, this means that M' will have a zero entry not covered by λ . If the site (i, j) is *doubly covered* by λ , that is, $i \in X$ and $j \in Y$, then we let $M'(i, j) = M(i, j) + t$. Finally if (i, j) is covered by exactly one file in λ , we let $M'(i, j) = M(i, j)$. Notice that the entries of M' are nonnegative.

Lemma 4.4. *Let M' be the reduction of M by the optimal cover λ . A k -assignment which is optimal in M is also optimal in M' .*

Proof. Let t be the minimum of the entries in M that are not covered by λ . For $s < t$, let M_s be the matrix obtained from M by subtracting s from the non-covered entries and adding s to the doubly covered entries. Since M_s has no zero entries except those of M , it follows from Theorem 4.3 that every optimal k -assignment in M_s must intersect every file of λ . By continuity, it follows that there is some optimal k -assignment in M' that intersects every file in λ . All k -assignments that intersect every file of λ are affected in the same way by the reduction from M to M' , namely if μ is such a k -assignment, then $\text{cost}_\mu(M') = \text{cost}_\mu(M) - (k - |\lambda|)t$. Hence if μ is an optimal k -assignment in M , then μ is optimal also in M' . \square

From Lemma 4.1 and König's theorem we can deduce the following:

Lemma 4.5. *There is an optimal cover of Z containing every row that belongs to some optimal cover of Z , and similarly there is an optimal cover that contains every column that belongs to some optimal cover.*

These covers are called the *row-maximal* and the *column-maximal* optimal covers, respectively.

Proof. It follows immediately from König's theorem that a file belongs to an optimal cover of Z if and only if it intersects every maximal

independent subset of Z . Let λ be the set of rows that belong to some optimal cover of Z . Let $Z \setminus \lambda$ be the set of sites in Z that are not covered by λ . Let μ be a maximal independent subset of $Z \setminus \lambda$. Then by Lemma 4.1 there is a maximal independent subset μ' of Z which intersects every row that intersects μ . At the same time, μ' must intersect every row in λ . Therefore $\text{rank}(Z \setminus \lambda) = \text{rank}(\mu) = \text{rank}(Z) - |\lambda|$. Hence λ can be extended to an optimal cover of Z . \square

We want to be able to do induction over matrix reduction. Therefore we need the following lemma:

Lemma 4.6. *Let $M = M_0$ be a nonnegative m by n matrix, and let $k \leq \min(m, n)$. For $i \geq 0$, let M_{i+1} be the reduction of M_i by the column-maximal optimal cover of M_i . Then one of the matrices M_i has a zero cost k -assignment.*

Proof. Let Z_i be the set of sites where M_i has zeros. Let λ_i be the column-maximal optimal cover of Z_i . By König's theorem, Z_0 has an independent subset μ containing exactly one site in each file in λ_0 . Hence μ contains no site which is doubly covered by λ_0 . It follows that $\text{rank}(Z_1) \geq \text{rank}(Z_0)$. Suppose that $\text{rank}(Z_1) = \text{rank}(Z_0)$. Every column in λ_1 must belong to λ_0 . Consequently every row in λ_0 must belong to λ_1 . Now since M_1 has a zero which is not covered by λ_0 , there has to be a row in λ_1 which is not in λ_0 .

To sum up, in each step of the reduction process, either the rank of the set of zeros increases, or the number of rows in the column-maximal optimal cover increases. \square

A feature of matrix reduction that has been exploited in several papers [LW00, AS02] is that it keeps track of the cost of the optimal k -assignment. In fact, if t is as above and M reduces to M' , then $\min_k(M) = (k - |\lambda|) \cdot t + \min_k(M')$. This means that we can compute $\min_k(M)$ recursively by iterating the reduction and keeping track of the values of t as well as the sizes of the optimal covers that are used. As long as the matrix entries are independent exponential variables, it is easy to compute the expected value of the minimum t , even for general m and n . However, since the doubly covered entries will eventually consist of sums of several dependent random variables, it becomes extremely hard to reach any conclusions valid for general k through this approach.

One of the key insights that led to the proof of the Parisi formula in [LW03] was the fact that information about the probability that a certain matrix element participates in the optimal assignment will give information about the expected minimum cost. However, a problem with the reduction algorithm is that in general, it loses track of the location of the optimal assignment.

Example 4.7. Here $k = 2$, and after the final step, the matrix contains two zero-cost 2-assignments, of which only one was optimal in the original matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 3 \end{pmatrix} \xrightarrow{\emptyset} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 2 \end{pmatrix} \xrightarrow{\{\text{column1}\}} \begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix} \xrightarrow{\{\text{row1}\}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The approach taken in this paper builds on an observation that has largely been overlooked, even in [LW03], namely that when the column-maximal optimal cover is used, matrix reduction keeps track of the set of rows that intersect the optimal k -assignment.

5. THE PARTICIPATION PROBABILITY LEMMA

In this section we prove a slightly refined version of a lemma which first occurred in [LW00]. This lemma describes the probability that a certain exponential variable participates in the optimal solution to a random assignment problem.

Lemma 5.1 ([LW00]). *Let M be a random matrix where a particular entry $M(i, j) \sim \exp(\alpha)$ is independent of the other matrix entries. Let M' be as M except that $M'(i, j) = 0$. Then the probability that (i, j) belongs to the optimal k -assignment in M is*

$$\alpha \cdot (\mathbb{E}[\min_k(M)] - \mathbb{E}[\min_k(M')]).$$

Proof. We condition on all entries in M except $M(i, j)$. Let M_t be the deterministic matrix obtained by also conditioning on $M(i, j) = t$. Let $f(t) = \min_k(M_t)$. Then either f is constant, or f increases linearly up to a certain point after which it is constant. The key observation is that the site (i, j) belongs to the optimal k -assignment in M_t if and only if $f'(t) = 1$ (disregarding the possibility that t is equal to the point where f is not differentiable). Therefore if $x \sim \exp(\alpha)$, then the probability that (i, j) belongs to the optimal k -assignment in M_x is equal to $\mathbb{E}[f'(x)]$. By partial integration we have

$$\begin{aligned} (6) \quad \mathbb{E}[f'(x)] &= \alpha \int_0^\infty e^{-\alpha t} f'(t) dt = \\ &= \alpha \int_0^\infty d(e^{-\alpha t} f(t)) + \alpha^2 \int_0^\infty e^{-\alpha t} f(t) dt \\ &= -\alpha f(0) + \alpha \mathbb{E}[f(x)] = \alpha \cdot (\mathbb{E}[\min_k(M)] - \mathbb{E}[\min_k(M')]). \end{aligned}$$

□

6. THE BUCK-CHAN-ROBBINS URN MODEL

The following urn model is described in [BCR02]: An urn contains a set of balls, each with a given positive weight. Balls are drawn one at a time without replacement, and each time the probability of drawing

a particular ball is proportional to the weight of the ball. This simple model has perhaps been studied before, but the connection to random assignment problems is due to Buck, Chan and Robbins.

To each weighted set we can associate such an urn process. Here we take as our weighted set the set R of row indices (in order not to make any secret of the kind of application we have in mind). We consider a continuous time version of this process. Each ball (row) i remains in the urn for an amount of time which is $\exp(w_R(i))$ -distributed, and the times at which the balls leave the urn are all independent.

The urn process is described by a continuous time random walk $u_R : \mathbf{R}^+ \rightarrow 2^R$ on the power set of R . For $t \geq 0$, $u_R(t)$ is the set of balls that have been drawn at time t .

If $X \subseteq R$, we denote by $Pr_R(X)$ the probability that this random walk reaches X , that is, the probability that every ball in X is drawn before every ball not in X .

Example 6.1. If there are three balls labeled 1, 2, 3 then

$$Pr_R(\emptyset) = Pr_R(\{1, 2, 3\}) = 1,$$

$$Pr_R(\{1\}) = \frac{w_R(1)}{w_R(\{1, 2, 3\})}$$

and

$$Pr_R(\{1, 2\}) = \frac{w_R(1)w_R(2)}{w_R(\{1, 2, 3\})w_R(\{2, 3\})} + \frac{w_R(1)w_R(2)}{w_R(\{1, 2, 3\})w_R(\{1, 3\})},$$

since the set $\{1, 2\}$ can be obtained either by first choosing 1 and then 2, or the other way around.

By an *order ideal* (or just ideal for short) we mean a family of sets of balls which is closed under taking subsets.

Lemma 6.2. *Suppose that I is an order ideal and further that $\emptyset \in I$ and $R \notin I$. Then*

$$(7) \quad \sum_{X \in I} \sum_{\substack{i \\ X \cup \{i\} \notin I}} \frac{w_R(i)Pr_R(X)}{w_R(\overline{X})} = 1.$$

Proof. The random walk u_R starts in \emptyset which is in I , and ends in R which is not in I . Since I is an order ideal, there will be exactly one step of the walk which leads from a set in I to a set not in I . The left hand side of (7) sums the probabilities of leaving I via a certain step, taken over all possible ways of leaving I . \square

If I is an ideal, and $R \notin I$, then we let $\text{exit}_R(I)$ be the random subset of R which is the first set in the urn process which does not belong to I . For $i \in R$, we denote by $I \setminus i$ the ideal consisting of all sets in I which do not contain i .

Lemma 6.3. *If $i \in R$ then*

$$Pr(i \in \text{exit}_R(I)) = \sum_{X \in I \setminus i} \frac{w_R(i) Pr_R(X)}{w_R(\overline{X})}.$$

Proof. $i \in \text{exit}_R(I)$ if and only if the ball i is drawn at a moment where the set of balls already drawn belongs to I . Therefore we get the probability of this event by summing the probability of first arriving at X and then drawing i in the next step, over all sets X in $I \setminus i$. \square

We can also describe this probability inductively in terms of the corresponding probability for smaller ideals. If I is an ideal and $i \in R$, we let $I/i = \{X \subseteq R : X \cup \{i\} \in I\}$.

Lemma 6.4. *If I is an ideal such that $\emptyset \in I$ and $R \notin I$, and $i_0 \in R$, then*

$$Pr(i_0 \in \text{exit}_R(I)) = \frac{w_R(i_0)}{w_R(R)} + \frac{1}{w_R(R)} \sum_{i \neq i_0} w_R(i) Pr(i_0 \in \text{exit}_R(I/i)).$$

Proof. The first term in the right hand side is the probability that i_0 is the first ball to be drawn. The probability that ball i is the first ball to be drawn is $w_R(i)/w_R(R)$, and given that this is the case, the ball i_0 belongs to $\text{exit}_R(I)$ if and only if it belongs to $\text{exit}_R(I/i)$. \square

If I is an ideal, we let $T_R(I) = \inf(t : u_R(t) \notin I)$ denote the *exit time* of I , in other words the time at which the random walk u_R leaves I , or equivalently the amount of time it spends in I . The following formula for the expected value $E[T_R(I)]$ of the exit time follows from the observation that the amount of time spent in I is equal to the sum of the time spent at each $X \in I$.

Lemma 6.5.

$$E[T_R(I)] = \sum_{X \in I} \frac{Pr_R(X)}{w_R(\overline{X})}.$$

Proof. Given that the walk reaches X , the expected amount of time until another ball is drawn is equal to

$$\frac{1}{w_R(\overline{X})}.$$

\square

7. A FORMULA FOR THE PARTICIPATION PROBABILITY OF A ROW

In this section we obtain a connection between the random assignment problem and the urn model by deriving a formula for the probability that a certain row intersects an optimal k -assignment. The special case of matrices without zero entries was proved in [BCR02]. Another special case, that of row- and column-weights equal to 1 (rate 1 exponential variables) was proved in [LW03] by a different method.

If M is a nonnegative random matrix, we let $\rho_k(M)$ be the (random) set of rows that intersect some optimal k -assignment in M . If Z is a set of sites, we let $r(Z)$ be the set of rows in the row-maximal optimal cover of Z . Moreover, if k is a positive integer, we let $I_k(Z)$ be the ideal of all sets of rows which are partial $k - 1$ -covers of Z .

Lemma 7.1. $X \in I_k(Z)$ iff $X \cup r(Z) \in I_k(Z)$.

Proof. We prove this by induction on the size of X . The induction step is equivalent to proving that the statement holds when X consists of one row, say $X = \{i\}$. If $i \in r(Z)$, then $X = X \cup r(Z)$, so the statement is obvious. If $i \notin r(Z)$, then let Z' be the set of sites in Z which are not in row i . By König's theorem, $\text{rank}(Z') = \text{rank}(Z)$. Hence an optimal cover of Z , in particular the row-maximal one, is also an optimal cover of Z' . \square

Corollary 7.2. Let Z be a set of sites, and let (i, j) be a site such that $\text{rank}(Z \cup \{(i, j)\}) = \text{rank}(Z) + 1$. Then $I_k(Z \cup \{(i, j)\}) = I_k(Z)/i$.

Proof. Since $i \in r(Z \cup \{(i, j)\})$, this follows from Lemma 7.1. \square

The following theorem establishes the connection between the urn process and the random assignment problem. Thereby it forms the basis for our approach, and in a sense it is the central theorem in the paper.

Theorem 7.3. *Let M be an m by n random matrix indexed by weighted sets R and C . Suppose that M has the following properties: There is a specified set Z of sites where the entries in M are zero. The remaining entries in the rows in $r(Z)$ are positive real numbers. For $i \in R$, $j \in C$, if $i \notin r(Z)$ and $(i, j) \notin Z$, then $M(i, j)$ is $\exp(w_R(i)w_C(j))$ -distributed and independent of the other matrix entries. Suppose that a certain row i_0 has no zeros. Then*

$$\Pr(i_0 \in \rho_k(M)) = \Pr(i_0 \in \text{exit}_R(I_k(Z))).$$

In our applications of this theorem, we are always dealing with standard matrices. However, to make the inductive proof go through, we must condition on the values of the nonzero entries in the rows in $r(Z)$. For this reason we let these entries be fixed numbers instead of random variables.

Proof. We prove this by induction. Let λ be the column-maximal optimal cover of Z . Let M' be the reduction of M by λ . Then there is at least one new zero in M' , that is, a site (i, j) which is not covered by λ and such that $M'(i, j) = 0$. We let $Z' = \{(i, j) : M'(i, j) = 0\}$. We consider two cases.

(1) All new zeros are in rows that belong to $r(Z)$. In this case $\text{rank}(Z') = \text{rank}(Z)$, and consequently $r(Z') = r(Z)$. It follows immediately from Lemma 7.1 that $I_k(Z') = I_k(Z)$. Hence by induction, $\Pr(i_0 \in \rho_k(M)) = \Pr(i_0 \in \text{exit}_R(I_k(Z)))$.

(2) There is a new zero $M'(i, j)$ such that $i \notin r(Z)$. Since $M(i, j)$ has continuous distribution and is independent of all other matrix entries, we may assume that $M'(i, j)$ is the only new zero in M' . Since the site (i, j) is not covered by any optimal cover of Z , we have $\text{rank}(Z') = 1 + \text{rank}(Z)$. Hence $i \in r(Z') = r(Z) \cup \{i\}$.

If $i = i_0$, then every optimal k -assignment in M' must intersect row i_0 . Since every optimal k -assignment in M is optimal in M' , every optimal k -assignment in M must intersect row i_0 . If on the other hand $i \neq i_0$, then with probability 1, row i_0 participates either in all or in none of the optimal k -assignments in M' . If we condition on the values of M' in row i , then M' satisfies the criteria of the theorem. Hence by induction, $Pr(i_0 \in \rho_k(M')) = Pr(i_0 \in \text{exit}_R(I_k(Z')))$. By Corollary 7.2 we have $I_k(Z') = I_k(Z)/i$. Therefore if we condition only on being in case 2, then

$$(8) \quad Pr(i_0 \in \rho_k(M)) = \frac{w_R(i_0)}{w_R(r(Z))} + \frac{1}{w_R(r(Z))} \sum_{\substack{i \notin r(Z) \\ i \neq i_0}} w_R(i) Pr(i_0 \in \text{exit}_R(I_k(Z)/i)),$$

which by Lemma 6.4 is equal to $Pr(i_0 \in \text{exit}_R(I_k(Z)))$. \square

8. THE TWO-DIMENSIONAL URN-PROCESS

At this point we introduce a kind of product of two urn processes. We consider two independent urn processes on the weighted sets R and C respectively. What we here call the *two-dimensional urn-process* is just a piece of notation that makes it easy to state the generalization of the Buck-Chan Robbins formula. The weight function is multiplicative: If $X \subseteq R$ and $Y \subseteq C$, then we let

$$w_{R \times C}(X, Y) = w_R(X)w_C(Y).$$

Time is two-dimensional, and we let

$$u_{R \times C}(x, y) = (u_R(x), u_C(y)).$$

We further let

$$Pr_{R \times C}(X, Y) = Pr_R(X)Pr_C(Y).$$

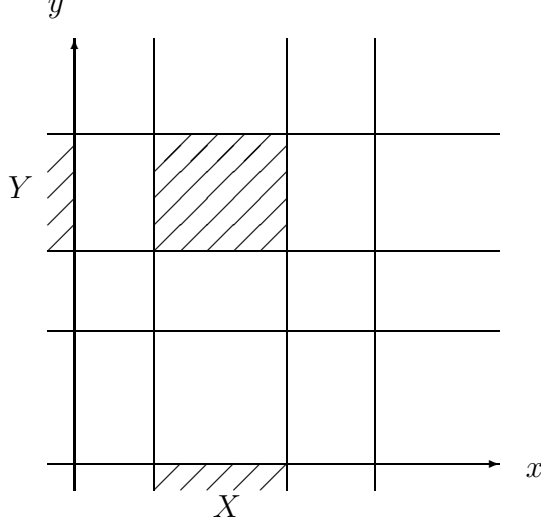
Since the two one-dimensional processes are statistically independent, $Pr_{R \times C}(X, Y)$ is equal to the probability that there exists a point (x, y) in the time plane such that $u_{R \times C}(x, y) = (X, Y)$.

Let J be an order ideal in $2^R \times 2^C$. In analogy with the one-dimensional exit time, we define the two-dimensional exit time $T_{R \times C}(J)$ to be the amount of two-dimensional time spent in J , that is, the area

of the region given by $u_{R \times C}(x, y) \in J$. We have

$$(9) \quad \mathbb{E}[T_{R \times C}(J)] = \sum_{(X, Y) \in J} \frac{Pr_{R \times C}(X, Y)}{w_{R \times C}(\overline{X}, \overline{Y})}.$$

As indicated in the figure, given that the random process reaches (X, Y) , the expected amount of time spent there is $1/w_{R \times C}(\overline{X}, \overline{Y})$.



9. A FORMULA FOR $\mathbb{E}[\min_k(M)]$

Let M be a standard matrix with rows and columns indexed by the weighted sets R and C . Let $J_k(M)$ be the ideal consisting of all partial $k-1$ -covers of the zeros of M . In this section we show that the expected cost of the minimal k -assignment in M is simply equal to the expected exit time of $J_k(M)$ in the two-dimensional urn process. When M is a matrix, we will write $I_k(M)$ for $I_k(Z)$, where Z is the set of zeros of M .

Theorem 9.1 (Main Theorem, probabilistic version).

$$(10) \quad \mathbb{E}[\min_k(M)] = \mathbb{E}[T_{R \times C}(J_k(M))].$$

The proof of Theorem 9.1 is inductive. We first prove that (10) is consistent with the row participation formula.

Lemma 9.2. *Let M be a standard matrix where row i_0 contains no zeros. Let M^j be obtained from M by setting the entry in position (i_0, j) equal to zero. If $\mathbb{E}[\min_k(M^j)] = \mathbb{E}[T_{R \times C}(J_k(M^j))]$ for every j , then $\mathbb{E}[\min_k(M)] = \mathbb{E}[T_{R \times C}(J_k(M))]$.*

Proof. By Lemma 5.1, the probability that the site (i_0, j) belongs to the optimal k -assignment in M is

$$w_R(i_0)w_C(j)(\mathbb{E}[\min_k(M)] - \mathbb{E}[\min_k(M^j)]).$$

Therefore, summing over j ,

$$w_R(i_0) \sum_{j \in C} w_C(j) (\mathbb{E}[\min_k(M)] - \mathbb{E}[\min_k(M^j)]) = \Pr(i_0 \in \rho_k(M)).$$

We divide by $w_R(i_0)$ and use the fact that by Lemma 6.3,

$$\Pr(i_0 \in \rho_k(M)) = \Pr(i_0 \in \text{exit}_R(I_k(M))) = \sum_{X \in I_k(M) \setminus i_0} \frac{w_R(i_0) \Pr_R(X)}{w(\bar{X})}.$$

Hence we obtain

$$(11) \quad \sum_{j \in C} w_C(j) (\mathbb{E}[\min_k(M)] - \mathbb{E}[\min_k(M^j)]) = \sum_{X \in I_k(M) \setminus i_0} \frac{\Pr_R(X)}{w(\bar{X})}.$$

It is clear that we can solve for $\mathbb{E}[\min_k(M)]$ in (11). To finish the induction step, it is therefore sufficient to prove that

$$(12) \quad \sum_{j \in C} w_C(j) (\mathbb{E}[T_{R \times C}(J_k(M))] - \mathbb{E}[T_{R \times C}(J_k(M^j))]) = \sum_{X \in I_k(M) \setminus i_0} \frac{\Pr_R(X)}{w_R(\bar{X})}.$$

If we fix a set $X \in I_k(M) \setminus i_0$, then by Lemma 6.2, applied to the ideal $\{Y : (X, Y) \in J_k(M)\}$, we have

$$\sum_{\substack{Y \\ (X, Y) \in J_k(M)}} \sum_{\substack{j \\ (X, Y \cup \{j\}) \notin J_k(M)}} \frac{w_C(j) \Pr_C(Y)}{w_C(\bar{Y})} = 1.$$

Since (i_0, j) is the only zero of M^j in row i_0 , a set of files not containing row i_0 can be extended to a $k - 1$ -cover of the zeros of M^j if and only if this can be done while making use of column j . If we want to cover the zeros of M^j as efficiently as possible, there is no point in using row i_0 if instead we can use column j . Therefore the condition $(X, Y \cup \{j\}) \notin J_k(M)$ on j in the inner sum can be replaced by $(X, Y) \notin J_k(M^j)$.

If we multiply by $\Pr_R(X)/w_R(\bar{X})$ and sum over all $X \in I_k(M) \setminus i_0$, we see that the right hand side of (12) equals

$$\sum_{X \in I_k(M) \setminus i_0} \sum_{\substack{Y \\ (X, Y) \in J_k(M)}} \sum_{\substack{j \\ (X, Y) \notin J_k(M^j)}} \frac{w_C(j) \Pr_R(X) \Pr_C(Y)}{w_R(\bar{X}) w_C(\bar{Y})}.$$

Here we can drop the conditions on X , since the inner sum will be empty unless $X \in I_k(N) \setminus i_0$. After changing the order of summation

so that the sum over j becomes the outer sum, this is equal to

$$\sum_{j \in C} w_C(j) \sum_{\substack{(X,Y) \in J_k(M) \\ (X,Y) \notin J_k(M^j)}} \frac{Pr_{R \times C}(X, Y)}{w_{R \times C}(\overline{X}, \overline{Y})}.$$

By (9), this is equal to the left hand side of (12). \square

Secondly, we show that (10) is consistent with removing a column that contains at least k zeros.

Lemma 9.3. *Suppose that $E[\min_{k-1}(M)] = E[T_{R \times C}(J_{k-1}(M))]$ for every standard matrix M , in other words, suppose that (10) holds when k is replaced by $k - 1$. Let M be a standard matrix that has a column with at least k zeros. Then $E[\min_k(M)] = E[T_{R \times C}(J_k(M))]$.*

Proof. Suppose that column j_0 , has at least k zeros. Let M' be the m by $n - 1$ matrix obtained by deleting the column j_0 of M . Since every $k - 1$ -assignment in M' can be extended to a k -assignment in M by including a zero in column j_0 , we have $E[\min_k(M)] = E[\min_{k-1}(M')]$. To prove the lemma, we therefore show that with the obvious coupling of the urn processes, $T_{R \times C}(J_k(M)) = T_{R \times C \setminus \{j_0\}}(J_{k-1}(M'))$.

Since there are k zeros in column j_0 , every $k - 1$ -cover of M must include j_0 . Therefore $(X, Y) \in J_k(M)$ if and only if $(X, Y \setminus \{j_0\}) \in J_{k-1}(M')$. The lemma follows. \square

We are now in a position to prove that (10) holds whenever m is sufficiently large compared to k .

Theorem 9.4. *If M is a standard m by n matrix with $m > (k - 1)^2$, then $E[\min_k(M)] = E[T_{R \times C}(J_k(M))]$.*

Proof. By Lemmas 9.2 and 9.3, it is sufficient to prove that the statement holds when M has at least one zero in each row, and no column with k or more zeros. In this case each column can contain the leftmost zero of at most $k - 1$ rows. Since there are more than $(k - 1)^2$ rows, there must be at least k columns that contain the leftmost zero of some row. This implies that there is a zero cost k -assignment in M . Consequently there is no $k - 1$ -cover, that is, $J_k(M) = \emptyset$. Plainly $E[\min_k(M)] = 0 = T_{R \times C}(\emptyset) = E[T_{R \times C}(J_k(M))]$. \square

Finally we prove that (10) holds also for smaller matrices by taking the limit as the weights of the exceeding rows tend to zero.

Proof of Theorem 9.1. We prove (10) by downwards induction on the number of rows. Suppose that M is a standard m by n matrix. Let M_ϵ be an augmented matrix of $m + 1$ rows and n columns, so that the first m rows equal M , and row $m + 1$ has no zeros and weight $w_R(m + 1) = \epsilon$. When ϵ is small, the entries of row $m + 1$ are large.

With high probability, none of them will participate in the optimal k -assignment, and consequently

$$\mathbb{E}[\min_k(M)] = \lim_{\epsilon \rightarrow 0} \mathbb{E}[\min_k(M_\epsilon)].$$

We therefore have to show that

$$\mathbb{E}[T_{R \times C}(J_k(M))] = \lim_{\epsilon \rightarrow 0} \mathbb{E}[T_{R \times C}(J_k(M_\epsilon))].$$

Since row $m + 1$ does not have any zero, we have $T_{R \times C}(J_k(M_\epsilon)) \leq T_{R \times C}(J_k(M))$ under the obvious coupling of the corresponding urn processes. Hence we can squeeze $\mathbb{E}[T_{R \times C}(J_k(M_\epsilon))]$ by

$$(1 - p)\mathbb{E}[T_{R \times C}(J_k(M))] \leq \mathbb{E}[T_{R \times C}(J_k(M_\epsilon))] \leq \mathbb{E}[T_{R \times C}(J_k(M))],$$

where p is the probability that there is a point (x, y) in the time plane such that $u_R(x), u_C(y)$ is a partial $k - 1$ -cover for M but not for M_ϵ . The only way we can have a partial $k - 1$ cover for M and not for M_ϵ is if the row $m + 1$ has been drawn. However, as $\epsilon \rightarrow 0$, the probability that row $m + 1$ is drawn among the first $k - 1$ rows goes to zero. Hence as ϵ tends to zero, so does p . The theorem follows. \square

10. PROOF OF THE BUCK-CHAN-ROBBINS FORMULA

We prove that the two formulations of the main theorem are indeed equivalent. We have

$$(13) \quad \sum_{(X,Y) \in J_k(M)} \frac{-\mu((X,Y), \hat{1})}{w_R(\overline{X})w_C(\overline{Y})} = \sum_{(X,Y) \in J_k(M)} (-\mu((X,Y), \hat{1})) \cdot \mathbb{E}[T_R(\{X' \subseteq X\})] \cdot \mathbb{E}[T_C(\{Y' \subseteq Y\})],$$

since the factors

$$\frac{1}{w_R(\overline{X})}, \quad \frac{1}{w_C(\overline{Y})}$$

can be interpreted as the expected exit times of the ideals $\{X' \subseteq X\}$ and $\{Y' \subseteq Y\}$ of all subsets of X and Y respectively. By Lemma 6.5,

$$(14) \quad \sum_{(X,Y) \in J_k(M)} \frac{-\mu((X,Y), \hat{1})}{w_R(\overline{X})w_C(\overline{Y})} = \sum_{(X,Y) \in J_k(M)} -\mu((X,Y), \hat{1}) \sum_{X' \subseteq X} \frac{Pr_R(X')}{w_R(\overline{X'})} \sum_{Y' \subseteq Y} \frac{Pr_C(Y')}{w_C(\overline{Y'})}$$

We now change the order of summation and get

$$(15) \quad \sum_{(X',Y') \in J_k(M)} \frac{Pr_R(X')Pr_C(Y')}{w_R(\overline{X'})w_C(\overline{Y'})} \sum_{(X',Y') \leq (X,Y) < \hat{1}} -\mu((X,Y), \hat{1}).$$

In the factor to the right, we are summing over all (X, Y) that satisfy $(X', Y') \leq (X, Y) < \hat{1}$ in the poset $\hat{J}_k(M)$. By the definition of the Möbius function, this sum is equal to 1, so that we can drop this factor. Hence (15) is equal to

$$\sum_{(X', Y') \in J_k(M)} \frac{Pr_R(X') Pr_C(Y')}{w(\overline{X'}) w(\overline{Y'})} = E[T_{R \times C}(J_k(M))].$$

This specializes to the equivalence of the two formulations of the main conjectures in [LW00] and of the two formulations in [BCR02]. The more general setting here allows us to give a shorter proof.

To finish the proof of Theorem 2.1, we cite a well-known theorem (G-C Rota?) that states that the Möbius function of the truncated Boolean lattice is indeed given by the binomial coefficient occurring in (3).

Theorem 10.1. *Let P be a poset consisting of the elements of rank $0, \dots, k-1$ in a Boolean lattice of degree N , together with an artificial top element $\hat{1}$. Then*

$$\mu(\emptyset, \hat{1}) = (-1)^k \binom{N-1}{k-1}.$$

Sketch of proof. Every interval of the form (\emptyset, x) for $x \neq \hat{1}$ is Boolean, and therefore $\mu(\emptyset, x) = (-1)^{\text{rank}(x)}$. Hence

$$\mu(\emptyset, \hat{1}) = - \sum_{\emptyset \leq x < \hat{1}} (-1)^{\text{rank}(x)} = - \sum_{i=0}^{k-1} (-1)^i \binom{N}{i}.$$

The theorem now follows immediately by induction on k . \square

11. SPECIALIZING TO RATE 1 VARIABLES

In this section we briefly comment on the implications of Theorem 9.1 to the case of rate 1 variables. In particular, we show that the Coppersmith-Sorkin formula follows. When there are no zeros in the matrix, the ideal $J_k(M)$ consists of all sets of at most $k-1$ files. Moreover, when all files have the same weight, the files become indistinguishable in the urn process. The expected amount of time until the next ball is drawn depends only on the number of balls already drawn, and not on which particular balls have been drawn. Therefore we obtain the expected value of $T(J_k(M))$ by simply conditioning on every step in the urn process taking exactly its expected amount of time. The expected amount of two-dimensional time that the process spends at the point where exactly i rows and j columns have been drawn is equal to

$$\frac{1}{(m-i)(n-j)}.$$

From this, the Coppersmith-Sorkin formula follows.

As $n \rightarrow \infty$, the process can be approximated by continuous exponential decay. At time t , a fraction of e^{-t} of the balls will remain in the urn. In the two-dimensional process, the borderline at which the process exits the ideal $J_n(M)$ will approach the curve given by the equation

$$e^{-x} + e^{-y} = 1.$$

Hence we may obtain the limit value as the area under this curve, which is indeed equal to $\zeta(2)$.

REFERENCES

- [A92] Aldous, David, *Asymptotics in the random assignment problem*, Pr. Th. Related Fields, **93** (1992) 507–534.
- [A01] Aldous, David, *The $\zeta(2)$ limit in the random assignment problem*, Random Structures Algorithms **18** (2001), no 4. 381–418.
- [AS02] Alm, Sven Erick and Sorkin, Gregory B., *Exact expectations and distributions in the random assignment problem*, Combin. Probab. Comput. **11** (2002), no. 3, 217–248.
- [BFM98] Beveridge Andrew, Frieze, Alan M. and McDiarmid, Colin J. H., *Random minimum length spanning trees in regular graphs*, Combinatorica **18** (1998) 311–333.
- [BCR02] Buck, Marshall W., Chan, Clara S., Robbins, David P., *On the expected value of the minimum assignment*, Random Structures & Algorithms **21** (2002), no. 1, 33–58.
- [CS98] Coppersmith, Don, Sorkin, Gregory B., *Constructive Bounds and Exact Expectations For the Random Assignment Problem*, Random Structures & Algorithms **15** (1999), 133–144.
- [CS02] Coppersmith, Don, Sorkin, Gregory B., *On the expected incremental cost of a minimum assignment*. In Contemporary Mathematics (B. Bollobás, ed.), Vol. 10 of *Bolyai Society Mathematical Studies*, Springer.
- [D69] Donath, W. E., *Algorithm and average-value bounds for assignment problems*, IBM J. Res. Dev., **13** (1969) 380–386.
- [EES01] Eriksson, Henrik, Eriksson, Kimmo, and Sjöstrand, Jonas, *Exact expectation for random graphs and assignments*, Proceedings of FPSAC 2001, Arizona.
- [FM89] Frieze, Alan M. and McDarmid Colin J.H., *On random minimum spanning trees*, Combinatorica **9** (1989).
- [F85] Frieze, Alan M., *On the value of a random minimum spanning tree problem*, Disc Applied Math. **10** (1985) 47–56.
- [GK93] Goemans, M. X., and Kodialam, M. S., *A lower bound on the expected cost of an optimal assignment*, Math. Oper. Res., **18** (1993), 267–274.
- [K87] Karp, R. M., *An upper bound on the expected cost of an optimal assignment*, In Discrete Algorithms and Complexity: Proceedings of the Japan-U.S. Joint Seminar, Academic Press, 1987, 1–4.
- [L93] Lazarus, A., *Certain expected values in the random assignment problem*, Oper. Res. Lett., **14** (1993), 207–214.
- [LW00] Linusson, Svante and Wästlund, Johan, *A generalization of the random assignment problem*, arXiv:math.CO/0006146. unpublished preprint.
- [LW03] Linusson, S. and Wästlund, J., *A proof of Parisi’s conjecture on the random assignment problem*, arXiv:math.CO/0303214.
- [LP86] Lovász, L. , Plummer, M. D., *Matching Theory*, North-Holland 1986.

- [MP85] Mézard, Marc and Parisi, Giorgio *Replicas and optimization*, J. Phys. Lett. **46**(1985), 771–778.
- [MP87] Mézard, Marc and Parisi, Giorgio, *On the solution of the random link matching problems*, J. Phys. Lett. **48** (1987), 1451–1459.
- [N02] Nair, Chandra, *Towards the resolution of Coppersmith-Sorkin conjectures*. Proceedings of the 40th annual Allerton conference on communication, control and computing, 2002.
- [NPS03] Nair, Chandra, Prabhakar, Balaji and Sharma, Mayank, *A proof of the conjecture due to Parisi for the finite random assignment problem*, available at <http://www.stanford.edu/~balaji/rap.html>.
- [O92] Olin, Birgitta, *Asymptotic properties of the random assignment problem*, Ph.D. thesis, Kungl Tekniska Högskolan, Stockholm, Sweden, (1992).
- [PR93] Pardalos, P. M. and Ramakrishnan, K. G., *On the expected optimal value of random assignment problems: Experimental results and open questions*. Comput. Optim. Appl. **2** (1993), 261–271.
- [P98] Parisi, Giorgio, *A conjecture on random bipartite matching*, Physics e-Print archive, <http://xxx.lanl.gov/ps/cond-mat/9801176>, January 1998.
- [S] Stanley, Richard, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge University Press, Cambridge, 1996.
- [W79] Walkup, D. W., *On the expected value of a random assignment problem*, SIAM J. Comput., **8** (1979), 440–442.

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